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DIMENSIONAL ANALYSIS

1. Physical mathematics. We learned from the ancients—Pythagoras (who died ~ 497 B.C.), Galileo (1564–1642)—that **world structure admits of mathematical description**, "God is a mathematician." This non-obvious fact seems never to lose its deeply surprising and mysterious quality in the imaginations of even the greatest physicists,[1] and becomes the more surprising when one appreciates that

- Most attempts to comprehend natural events—historically, cross-culturally —have been phrased in qualitative language, the language of (say) myth.
- We are still obliged to use qualitative language when discussing most of the concepts (beauty, justice, . . .) and problems ("Why did she do that?") encountered in the course of our non-scientific day-to-day affairs. Physics is, in this respect, "special."
- Physical calculations, no matter how long/intricate/abstract they may become, manage somehow to "stay in touch with the world."

It is interesting to ask: **Can one imagine a world so unstructured/chaotic as NOT to admit of a "mathematical physics?"**

Numbers—actually integers, which are promoted to real numbers by acts of abstraction—enter physics (and the sciences generally) by

- **counting**, and that special kind of counting called
- **measurement**.

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[1] E. P. Wigner's essay "The unreasonable effectiveness of mathematics in the natural sciences," Comm. Pure Appl. Math. **13**, 1–14 (1960) is a classic in the field. See also R.W.Hamming, "The unreasonable effectiveness of mathematics," Amer. Math. Monthly **87**, No. 2 (February 1981), which can be found on the web at http://www.lecb.ncifcrf.gov/~toms/Hamming.unreasonable.html.

Interesting subtleties aside, it is, I think, the role assigned to measurement intruments that distinguishes the physical sciences from all other branches of knowledge—among them philosophy and mathematics itself. It was historically the search for patterns among physically generated numbers that inspired the invention of much fundamental mathematics. But by the beginning of the 19^{th} Century it had become clear that mathematics is an autonomous subject, related to but unconstrained by contingent world-structure. Thus arose the distinction between pure and applied mathematics. Physicists—when not acting as instrument builders, dreamers...—are applied mathematicians, mathematical model makers.

It is useful to attempt to clarify the pure/applied distinction. When a (pure) mathematician writes

$$f(x,\ddot{x};m,k) \equiv m\ddot{x} + kx = 0$$

he has described precisely a problem that lives only in his mind, the problem that he proposes to discuss. But when a physicist writes that same equation he has described precisely only how he proposes to *model the* (*approximate*!) *behavior of a class of objectively real physical systems*. For the physicist (but not for the mathematician) the variables (x,t) and the parameters (m,k) are considered to have *objective referents*; *i.e., to have* <u>*number values derived from acts of measurement*</u>. This circumstance gives the physicist a kind of "heuristic advantage" which is denied the pure mathematician, and which it will be my primary objective here to describe.

2. Measurement: units & dimensionality. For the simple purposes at hand it will be sufficient to consider by way of illustration the measurement of **length**. Similar remarks pertain, with interesting variations, to the measurement of mass/time/temperature...indeed, to all the variables/parameters that enter properly into physics, though it is not clear that they pertain with the same force to some of the variables ("propensities") contemplated by economists and psychologists.

Physics is circular. Before we can undertake the measurement of (say) length we must have some preliminary sense—however informal/tentative—of

1) what the concept in question (distance between two points) "means"
2) what operations can sensibly (which is to say: consistently with a physics yet to be invented!) relate to its "measurement."

Those bridges crossed (and they have from time to time to be reexamined), we

1) pick (arbitrarily/conveniently) a reproducible **"standard instance" of the concept in question** (inscribe two marks on a stable bar, these defining our **unit** of length) and
2) proceed operationally to **quantify natural instances of the concept by stating "how many times they contain the unit."** I need not describe how one uses a meter stick, but point out that

a) we proceed on the (usually tacit) assumption that meter sticks do not change their defining properties when transported from one spacetime region to another;

b) meter sticks differ profoundly from (say) clocks and thermometers, for one cannot "lay temporal (or thermal) intervals side by side" as one can and does lay a meter stick beside the object whose length one wants to measure;

c) to measure very short or very long spatial intervals one can/does *not* use a meter stick: the concept of length (which in well-established practice spans at least forty orders of magnitude) is quantified by a **spliced hierarchy of distinct operational procedures**.

Preceding remarks may serve to establish this point: metrology lives at the frontiers of both technology and philosophy. It may, at first blush, seem dull, but it is definitely not trivial. A science can be no more secure than its metrological foundations.

When we say that "the distance from A to B is about x" we mean that

1) we imagine there to exist a "true distance" having "the nature of a length," *i.e.*, the **dimensionality** of length;

2) measurement has or would yield the value

$$x \text{ length units}$$

where x is a real number, known only to within some **observational error** Δx (which may be irreducible-in-principle).

When we wish to indicate the dimensionality (as opposed to the numerical value) of x we write $[x]$: thus

$$[x] = \text{length}$$

It is clear that **units and dimensionality are quite distinct concepts**, not to be confused, certainly not interchangable. There are, as will emerge, subtle senses in which the concept of dimensionality is convention-dependent, but the conventionality of the unit is obvious/manifest. This is a circumstance unaffected by the fact that **mature physics supplies certain "natural" units** —thus

$$\begin{aligned} c &= \text{natural unit of velocity} \\ e &= \text{natural unit of electric charge} \\ \hbar &= \text{natural unit of action} \\ &\vdots \end{aligned}$$

—but these are seldom natural in the sense "most convenient for practical work."

3. Dimensional interrelationships. Expansion of the phenomenological scope of our physics tends to *increase* the number of what we may initially suppose to be independently dimensioned physical concepts (each measured in its own units), while the discovery of a (deep) physical law serves often to *decrease* the number of independently dimensional concepts, and to establish numerical interrelationships within our system of units. By way of clarification…

Suppose that—experts in length-measurement that we are—we expand our physics to include the concept of area. We measure lengths in (say) **centimeters**, but the new concept (area) requires the introduction of a corresponding new unit: we adopt (say) the **acre**. "Research" shows that the area A of any rectangle (sides of lengths x and y) can be described

$$A = \frac{1}{40468564}xy$$

This statement is, however, unacceptable as a statement of "natural law" because its **validity is contingent upon a convention** (choice of units). To avoid this formal defect we write (with A in acres, x and y in centimeters)

$$A = kxy \tag{1}$$

$$k = \frac{1}{40468564}\text{ acres/centimeter}^2$$

$$[k] = \text{area/length}^2 \tag{2}$$

The point is that (1) captures—in a convention-independent manner—the **structure of the functional relationship** among A, x and y. It is evident that by unit-adjustment we could in particular arrange to have

$$k = \mathbf{1}\ \text{(adjusted area unit)/(adjusted area unit)}^2$$

—this being a mere process-of-convenience that leaves (1) unchanged. Less evidently, we might insist that

$$k = \mathbf{1}\ \text{ is } \textbf{dimensionless}$$

which—by (2)—enforces

$$[\text{area}] = [\text{length}]^2$$

This is by nature a **dimensional interrrelation**, and means that we can discard as redundant our unit of area. The "composite nature of area" would, of course, be one of the central features of the "theory of area" which would be the final product of our "physical" research. If the above seems far-fetched and belabored, consider the following:

We are (let us suppose) interested in the dynamics of mass points. We discover the subject to be dominated by four key concepts—**mass, length, time, force**—which we quantify by introducing a corresponding quartet of units. Experimental research leads to the conclusion that

$$F = km\ddot{x}$$

$$k = \text{numeric}\cdot\frac{\text{(force unit)}}{\text{(mass unit)(length unit)/(time unit)}^2}$$

We may interpret the observed **universality** of $F = km\ddot{x}$ (*i.e.*, the fact that it is found to work with the same k for all forces F and all masses m) as an invitation to set

$$k = \mathbf{1} \quad \text{(dimensionless)}$$

Then

$$F = m\ddot{x} \tag{3}$$

gives

$$[\text{force}] = [\text{mass}][\text{length}][\text{time}]^{-2} \tag{4}$$

and **one of our units has been rendered redundant**: this is standardly read as license to write

$$\text{force unit} = (\text{mass unit})(\text{length unit})/(\text{time unit})^2 \tag{5}$$

Suppose now that we expand our interest to include the dynamics of *electrically charged* particles. A new concept—**charge**—and a corresponding new unit enter the discussion. Experimentally

$$\begin{aligned}
F &= k\frac{q_1 q_2}{r^2} \\
k &= \text{numeric}\cdot\frac{(\text{force unit})(\text{length unit})^2}{(\text{charge unit})^2} &(6)\\
&\equiv \frac{1}{4\pi\epsilon_0} = 8.99\times 10^9 (\text{Newton})(\text{meter})^2/(\text{Coulomb})^2 &(7)\\
&= \text{numeric}\cdot(\text{mass unit})(\text{length unit})^3(\text{time unit})^{-2}(\text{charge unit})^{-2}
\end{aligned}$$

Here again it becomes attractive to set

$$k = \mathbf{1} \quad \text{(dimensionless)}$$

Coulomb's law then assumes the form

$$F = \frac{q_1 q_2}{r^2} \tag{8}$$

giving

$$\begin{aligned}
[\text{charge}] &= [\text{force}]^{\frac{1}{2}}[\text{length}] \\
&= [\text{mass}]^{\frac{1}{2}}[\text{length}]^{\frac{3}{2}}[\text{time}]^{-1} &(9)
\end{aligned}$$

and again one of our units (taken most naturally to be the charge unit) has been rendered redundant. This option gives rise to the so-called "electrostatic system of units." Engineers prefer, however, to regard charge as a dimensionally independent entity; *i.e.*, to live with (6) and (7).

Suppose now that we expand our sphere of interest once again, to embrace the *gravitational* interaction of material particles. We might expect a new concept—“**gravitational charge**” μ—and a corresponding new unit to enter the picture. The essence of Newton’s Universal Law of Gravitation resides in the two-part assertion that

$$F_{\text{gravitation}} = K\frac{\mu_1\mu_2}{r^2} \tag{10.1}$$

$$\mu = km \quad \text{with } k \text{ the same for all bodies, all materials} \tag{10.2}$$

—assertions that when taken in combination yield the more commonly encountered form

$$\begin{aligned} F_{\text{gravitation}} &= G\,\frac{m_1m_2}{r^2} \\ G &= Kk^2 \end{aligned} \tag{11}$$

The discovered universality[2] of (10.2) entitles us to set $k = 1$ (dimensionless); *i.e.*, to **identify the concepts of gravitational charge and inertial mass**.[3] Working from (11) one has

$$\begin{aligned} G &= \text{numeric}\cdot\frac{(\text{force unit})(\text{length unit})^2}{(\text{mass unit})^2} \\ &= 6.672\times10^{-11}(\text{Newton})(\text{meter})^2/(\text{kilogram})^2 \\ &= \text{numeric}\cdot(\text{mass unit})^{-1}(\text{length unit})^3(\text{time unit})^{-2} \end{aligned}$$

Were we to yield now to our recently acquired instinct we might attempt to set

$$= 1 \quad (\text{dimensionless})$$

This, however, would require that the dimensions presently assigned to G be redistributed. Suppose we assume—arbitrarily—that that full responsibility is

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[2] Had Galileo actually performed the Leaning Tower experiment he would have been in position to assert, in about 1590, that

$$\kappa_{a/b} \equiv \frac{k_a}{k_b} = \frac{\mu_a/m_a}{\mu_b/m_b} = 1 \pm 10^{-2}$$

Newton (1686) obtained $\kappa_{a/b} = 1 \pm 10^{-3}$ from the physics of pendula, and by 1832 Bessel had obtained $\kappa_{a/b} = 1 \pm 10^{-5}$ by a refinement of Newton’s method. Such experiments are usually associated with the name of Baron Lórand von **Eötvös**, who in 1922 used a torsion balance technique to achieve $\kappa_{a/b} = 1 \pm 10^{-8}$. A lunar laser ranging experiment obtained $\kappa_{a/b} = 1 \pm 10^{-12}$ in 1976, and by 2003 an astronomical technique had given $\kappa_{a/b} = 1 \pm 10^{-18}$. This continuing experimental effort is motivated by the fact that the Principle of Equivalence (see below) is fundamental to general relativity.

[3] That $m_{\text{gravitational}} \equiv m_{\text{inertial}}$ is the upshot of the **principle of equivalence**.

to be assigned to [mass], and that [length] and [time] are to remain passive by-standers. If we insist that Newton's 2^{nd} law is to retain its familiar design then

$$m \longmapsto \tilde{m} = \alpha m \quad \text{enforces} \quad F \longmapsto \tilde{F} = \alpha F$$

This said, we observe that if we set $\alpha = G$ then

$$F = m\ddot{x} \quad \text{entails} \quad \tilde{F} = \tilde{m}\ddot{x}$$
$$F_{\text{gravitational}} = G\,\frac{m_1 m_2}{r^2} \quad \text{entails} \quad \tilde{F}_{\text{gravitational}} = \frac{\tilde{m}_1\tilde{m}_2}{r^2}$$

and the "gravitostatic mass" $\tilde{m}$ becomes dimensionally redundant with [length] and [time]:

$$[\text{gravitostatic mass}] = [\text{length}]^3[\text{time}]^{-2}$$

This, however, is not standardly done…for, I suppose, some mix of the following (mainly practical) reasons:

1) It seems inappropriate to build gravitation into the foundations of our metrology since
   a) gravitational forces are (relatively) so weak
   b) gravitational effects are irrelevant to most of our physics.
2) While F and m can be measured by well established laboratory procedures, $\tilde{F}$ and $\tilde{m}$ cannot be.
3) G is known with insufficient precision to make the program described above metrologically sound
4) It could be argued—I would argue—that **constants of nature**

$$\{G, \text{also } c, e, \hbar, \ldots\}$$

   are too important to be metrologically disguised, at least until physics has reached a point of higher maturity.
5) Mass is much too important/busy a concept to be assigned the awkward composite dimension stated above.

We conclude from the preceding discussion that in this dusty corner of physics one confronts—depending upon how much one knows about world-structure—many options, and that whether one elects to exploit those options is a question settled on grounds partly utilitarian (the nature of the job at hand) and partly conventional. *Definitions* (density $\equiv$ mass/volume, velocity $\equiv$ length/time) lead trivially to concepts of composite dimension, and to dimensional interrelationships of a trivial nature. On the other hand, *physical laws* ($F = ma$, $E = mc^2$, $E = h\nu$, …)—to the extent that they are "deep/fundamental—bring to light dimensional interrelationships that contain the element of surprise. All such interrelationships provide opportunities to contract the list of "primary dimensions (units)." The constants of Nature which remain explicit in our physical equations signal by-passed opportunities. The question **How many dimensions are primary?** (in the sense that from them can be assembled all others) has therefore a semi-conventional answer.

So much for what might be called the "philosophical" aspects of our topic. I turn now to discussion of its practical ramifications.

4. The principle of dimensional homogeneity. If, in the interest of concreteness, we *assume mass, length and time to be dimensionally primary* then for any physical variable X we find that measurement supplies a datum of the form

$$x\,(\text{mass units})^a(\text{length units})^b(\text{time units})^c \tag{12}$$

and to describe the evident dimensionality of X we write

$$[X] = M^a L^b T^c \tag{13}$$

Note that the meaning of (13) is independent of any particular choice of mass/length/time units, but that the value of the numeric x that assigns measured value to X depends critically upon such choices. Suppose, for example, we were to **change units** as follows:

$$\left.\begin{array}{c}\text{mass units} = A \text{ new mass units}\\ \text{length units} = B \text{ new length units}\\ \text{time units} = C \text{ new time units}\end{array}\right\} \tag{14}$$

where A, B and C are dimensionless numerics. The measured value of X then becomes

$$\bar{x}\,(\text{new mass units})^a(\text{new length units})^b(\text{new time units})^c$$

with

$$\bar{x} = x\,A^a B^b C^c \tag{15}$$

Suppose now that $x_1, x_2, \ldots, x_n$ refer (relative to some prescribed system of units) to the measured values of a set of variables[4] $X_1, X_2, \ldots, X_n$ that enter into the construction of some physical theory, and suppose further that

$$y = f(x_1, x_2, \ldots, x_n) \tag{16}$$

describes a physical relationship—a "formula"—encountered within that theory. Write

$$\left.\begin{array}{r}[\,Y\,] = M^a L^b T^c\\ [X_1] = M^{a_1} L^{b_1} T^{c_1}\\ [X_2] = M^{a_2} L^{b_2} T^{c_2}\\ \vdots\quad\\ [X_n] = M^{a_n} L^{b_n} T^{c_n}\end{array}\right\} \tag{17}$$

[4] By "variables" I here mean "parameters, variables, derivatives of variables with respect to variables, natural constants, etc."

It follows by (15) that a change of units (14) induces

$$\left.\begin{aligned} y &\longmapsto \bar{y} = y\ A^{a}\ B^{b}\ C^{c} \equiv K\ y \\ x_1 &\longmapsto \bar{x}_1 = x_1 A^{a_1} B^{b_1} C^{c_1} \equiv K_1 x_1 \\ x_2 &\longmapsto \bar{x}_2 = x_2 A^{a_2} B^{b_2} C^{c_2} \equiv K_2 x_2 \\ &\vdots \\ x_n &\longmapsto \bar{x}_n = x_n A^{a_n} B^{b_n} C^{c_n} \equiv K_n x_n \end{aligned}\right\} \tag{18}$$

Equations (18) describe what happens when we exercise our option to modify our metrological conventions. It is clear that statements—such as (16)—which refer to world-structure **must be stable under (18)**:

$$y = f(x_1, x_2, \dots, x_n) \quad \Longleftrightarrow \quad \bar{y} = f(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n)$$

This entails that

$$Kf(x_1, x_2, \dots, x_n) = f(K_1 x_1, K_2 x_2, \dots, K_n x_n) \tag{19}$$

must have the status of an **identity** in the variables $\{x_1, x_2, \dots, x_n, A, B, C\}$. Not all functions $f(\bullet)$ have this property. Those which do are said to be **dimensionally homogeneous**. The

> PRINCIPLE OF DIMENSIONAL HOMOGENEITY asserts that *only functions $f(\bullet)$ that are dimensionally homogeneous can figure properly in our physics.*

The principle derives its power and importance from the facts that it *restricts the class of physically acceptable $f(\bullet)$-functions*, provides a valuable check on the accuracy/plausibility of conjectured physical functions and often permits one to guess the form of physical functions even in advance of the development of a detailed theoretical account of the phenomenon in question. In this respect the principle of dimensional homogeneity resembles (say) the principle of Lorentz covariance: both speak formally to the **transformational properties** of physically admissible equations... and both serve to tell us "where the physics can't be," where it may plausibly be.

PROBLEM 1: Proceeding in the notation of (17), use (19) to show

a) that

$$y = x_1 + x_2 + \ldots + x_n$$

conforms to the principle of dimensional homogeneity if and only if

$$[Y] = [X_1] = [X_2] = \cdots = [X_n]$$

which is to say: if and only if $a_i = a$, $b_i = b$, $c_i = c$ (all i).

b) that

$$y = x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n} \tag{20.1}$$

conforms to the principle of dimensional homogeneity if and only if

$$\left.\begin{aligned} a_1k_1 + a_2k_2 + \cdots + a_nk_n &= a\\ b_1k_1 + b_2k_2 + \cdots + b_nk_n &= b\\ c_1k_1 + c_2k_2 + \cdots + c_nk_n &= c \end{aligned}\right\} \tag{20.2}$$

5. Simplest elements of dimensional analysis. Equations (20.2) might be notated

$$\underbrace{\begin{pmatrix} a_1 & a_2 & \dots & a_n\\ b_1 & b_2 & \cdots & b_n\\ c_1 & c_2 & \cdots & c_n \end{pmatrix}}_{3\times n \text{ matrix, call it } \mathbb{M}} \begin{pmatrix} k_1\\ k_2\\ \vdots\\ k_n \end{pmatrix} = \begin{pmatrix} a\\ b\\ c \end{pmatrix} \tag{21.1}$$

or again

$$\left.\begin{aligned} \boldsymbol{a}\cdot\boldsymbol{k} &= a\\ \boldsymbol{b}\cdot\boldsymbol{k} &= b\\ \boldsymbol{c}\cdot\boldsymbol{k} &= c \end{aligned}\right\} \tag{21.2}$$

where $\boldsymbol{k}$, $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ are n-dimensional column vectors

$$\boldsymbol{k} \equiv \begin{pmatrix} k_1\\ k_2\\ \vdots\\ k_n \end{pmatrix}, \quad \boldsymbol{a} \equiv \begin{pmatrix} a_1\\ a_2\\ \vdots\\ a_n \end{pmatrix}, \quad \boldsymbol{b} \equiv \begin{pmatrix} b_1\\ b_2\\ \vdots\\ b_n \end{pmatrix}, \quad \boldsymbol{c} \equiv \begin{pmatrix} c_1\\ c_2\\ \vdots\\ c_n \end{pmatrix}$$

and where the dot products have the familiar meaning:

$$\boldsymbol{a}\cdot\boldsymbol{k} \equiv a_1k_1 + a_2k_2 + \cdots + a_nk_n$$

The linear system (21.1) can be solved

$$\boldsymbol{k} = \mathbb{M}^{-1}\begin{pmatrix} a\\ b\\ c \end{pmatrix} \quad \text{provided} \quad \det\mathbb{M} \neq 0$$

in the case $n = 3$, but

- is **overdetermined** (more equations than unknown k's) if $n < 3$;
- is **underdetermined** (fewer equations than unknown k's) if $n > 3$.

EXAMPLE: Suppose we were to conjecture that

$$\text{force} = (\text{mass})^{k_1} \cdot (\text{velocity})^{k_2} \cdot (\text{acceleration})^{k_3}$$

Dimensionally we would have

$$M^1L^1T^{-2} = (M^1L^0T^0)^{k_1}(M^0L^1T^{-1})^{k_2}(M^0L^1T^{-2})^{k_3}$$

which in the notation of (21.1) becomes

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -2 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

and by matrix inversion gives

$$\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

whence

$$\begin{aligned} \text{force} &= (\text{mass})^1 \cdot (\text{velocity})^0 \cdot (\text{acceleration})^1 \\ &= (\text{mass}) \cdot (\text{acceleration}) \end{aligned}$$

A MORE INTERESTNG EXAMPLE: Suppose—with pendula on our minds—we were to conjecture that

$$\text{period} = (\text{mass of bob})^{k_1} \cdot (\text{length of rod})^{k_2} \cdot (g)^{k_3}$$

Dimensionally we would have

$$M^0 L^0 T^1 = (M^1 L^0 T^0)^{k_1} (M^0 L^1 T^0)^{k_2} (M^0 L^1 T^{-2})^{k_3}$$

which in the notation of (21.1) becomes

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and by matrix inversion gives

$$\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ +\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$$

whence

$$\text{period} = \sqrt{(\text{mass})^0 \cdot (\text{length})^1 / (\text{gravitational acceleration})^1}$$

Dimensional analysis has by itself informed us that (rather surprisingly) the period of a pendulum is independent of the mass of the bob. Detailed dynamical theory serves only to supply a dimensionless factor of 2π.

AN OVERDETERMINED SYSTEM: Suppose—with flapping flags in mind—we were to conjecture that

$$\text{period} = (\text{mass of flag})^{k_1}\cdot(\text{wind speed})^{k_2}$$

Dimensionally we would have

$$M^0L^0T^1 = (M^1L^0T^0)^{k_1}(M^0L^1T^{-1})^{k_2}$$

In the notation of (21.1) this becomes

$$\begin{pmatrix}1 & 0\\ 0 & 1\\ 0 & -1\end{pmatrix}\begin{pmatrix}k_1\\ k_2\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 1\end{pmatrix}$$

which quite clearly possesses <u>no solution</u>, the implication being that our conjecture is untenable.

AN UNDERDETERMINED SYSTEM: Suppose our pendulum is subject not only to gravitational but also to electrical forces. We conjecture that

$$\text{period} = (\text{mass})^{k_1}\cdot(\text{length})^{k_2}\cdot(\text{g})^{k_3}\cdot(\text{electrical force})^{k_4}$$

Dimensionally we would have

$$M^0L^0T^1 = (M^1L^0T^0)^{k_1}(M^0L^1T^0)^{k_2}(M^0L^1T^{-2})^{k_3}(M^1L^1T^{-2})^{k_4}$$

which in the notation of (21.1) becomes

$$\begin{pmatrix}1 & 0 & 0 & 1\\ 0 & 1 & 1 & 1\\ 0 & 0 & -2 & -2\end{pmatrix}\begin{pmatrix}k_1\\ k_2\\ k_3\\ k_4\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 1\end{pmatrix}$$

This linear system can be written

$$\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 1\\ 0 & 0 & -2\end{pmatrix}\begin{pmatrix}k_1\\ k_2\\ k_3\end{pmatrix} = \begin{pmatrix}-k_4\\ -k_4\\ 1+2k_4\end{pmatrix}$$

and gives

$$\begin{pmatrix}k_1\\ k_2\\ k_3\end{pmatrix} = \begin{pmatrix}0\\ +\frac{1}{2}\\ -\frac{1}{2}\end{pmatrix} - \begin{pmatrix}k_4\\ 0\\ k_4\end{pmatrix}$$

so we are led to a <u>one-parameter family</u> of possibilities:

$$\text{period} = (\text{mass})^{-u}\cdot(\text{length})^{\frac{1}{2}}\cdot(\text{g})^{-(\frac{1}{2}+u)}\cdot(\text{electrical force})^{u}$$

In the absence of electrical force (*i.e.*, at $u = 0$) we recover the result developed on the preceding page. In the presence of such a force we would have to look beyond dimensional analysis—to the detailed dynamics of the system—to fix the value of u.

6. Dimensionless products. Clearly, one will have

$$[y] = [y^{\text{power}}] \quad \text{if and only if } y \text{ is dimensionless}$$

And from this it follows that equations of (say) the form $z = e^y$, and more generally of the form

$$z = \text{power series in } y$$

are physically admissible (conform to the principle of dimensional homogeneity) if and only if $y(x_1, x_2, \ldots, x_n)$ has been assembled dimensionlessly from the variables/parameters/constants characteristic of the system under study. Thus do we acquire special interest in <u>dimensionless functions of the system variables</u>. That interest will soon be reenforced by quite another consideration.

It is evident that if c_1 and c_2 are dimensionless constants (things like $\frac{1}{2}$ and π) and if y_1 and y_2 are dimensionless physical variables, then so are $y_1 \cdot y_2$ and $c_1 y_1 + c_2 y_2$ dimensionless: *the set of all dimensionless constants/variables is closed under both addition and multiplication.* Elements of that set are standardly denoted Π, and have the **same numerical values in all systems of units**.

It follows directly from (21) that $\Pi = x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$ will be dimensionless if and only if

$$\underbrace{\begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \\ c_1 & c_2 & \cdots & c_n \end{pmatrix}}_{3\times n \text{ matrix, call it } \mathbb{M}} \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{22.1}$$

or more succinctly

$$\mathbb{M}\boldsymbol{k} = \boldsymbol{0} \tag{22.2}$$

Equivalently

$$\left.\begin{aligned} \boldsymbol{a}\cdot\boldsymbol{k} &= 0 \\ \boldsymbol{b}\cdot\boldsymbol{k} &= 0 \\ \boldsymbol{c}\cdot\boldsymbol{k} &= 0 \end{aligned}\right\} \tag{22.3}$$

The vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{k}$ live in an n-dimensional vector space where, according to (22.3), $\boldsymbol{k}$ stands normal to the sub-space spanned by $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$:

$$\boldsymbol{k} \perp \{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$$

How many linearly independent such $\boldsymbol{k}$-vectors are there? The obvious answer is

$$p = n - r, \text{ where} \begin{cases} r \text{ is the dimension (1 or 2 or 3) of} \\ \text{the sub-space spanned by } \{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\} \end{cases} \tag{23}$$

SIMPLE PENDULUM REVISITED: This familiar system presents the variables

$$
\begin{aligned}
\text{period } \tau \equiv x_1 &\quad : \quad [x_1] = M^0L^0T^1 \\
\text{bob mass } m \equiv x_2 &\quad : \quad [x_2] = M^1L^0T^0 \\
\text{rod length } \ell \equiv x_3 &\quad : \quad [x_3] = M^0L^1T^0 \\
\text{gravitational acceleration } g \equiv x_4 &\quad : \quad [x_4] = M^0L^1T^{-2}
\end{aligned}
$$

Equation (22.1) has become

$$
\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \text{ abbreviated } \mathbb{M}\boldsymbol{k} = \mathbf{0}
$$

Mathematica provides a command `NullSpace[`*rectangular matrix*`]` that provides a list of linearly independent solutions $\boldsymbol{k}_1, \boldsymbol{k}_2, \ldots, \boldsymbol{k}_p$ of the equation $\mathbb{M}\boldsymbol{k} = \mathbf{0}$. In the present instance it supplies a single solution:

$$
\boldsymbol{k} = \begin{pmatrix} 2 \\ 0 \\ -1 \\ 1 \end{pmatrix}
$$

Thus are we led to the dimensionless construct

$$
\Pi = \tau^2 m^0 \ell^{-1} g^1
$$

and from

$$
\Pi = \text{dimensionless constant}
$$

we recover

$$
\tau \sim \sqrt{\ell/g}
$$

CHARGED PENDULUM REVISITED: Some train of thought motivates us to add

$$
\text{electric force } \mathcal{E} \equiv x_5 \quad : \quad [x_5] = M^1L^1T^{-2}
$$

to our list of pendular variables. Equation (22.1) has now become

$$
\begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & -2 & -2 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \\ k_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

The `NullSpace`[*rectangular matrix*] command now provides *two* linearly independent solutions of $\mathbb{M}\boldsymbol{k} = \mathbf{0}$:

$$\boldsymbol{k}_1 = \begin{pmatrix} 2 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{k}_2 = \begin{pmatrix} 2 \\ -1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

Thus are we led to the dimensionless constructs

$$\begin{aligned} \Pi_1 &= \tau^2 m^0 \ell^{-1} g^1 \mathcal{E}^0 \\ \Pi_2 &= \tau^2 m^{-1} \ell^{-1} g^0 \mathcal{E}^1 \end{aligned}$$

On page 12 we were led by a different line of argument to a one-parameter family of dimensionless constructs

$$\Pi(u) = \tau^2 m^{2u} \ell^{-1} g^{2u+1} \mathcal{E}^{-2u}$$

It is interesting to note in this regard that

$$(\Pi_1)^p (\Pi_2)^q = \tau^{2(p+q)} m^{-q} \ell^{-(p+q)} g^p \mathcal{E}^q$$

is a two-parameter family of dimensionless constructs that gives back $\Pi(u)$ when we set $p = 1 + 2u$ and $q = -2u$.

GRAVITATIONAL PHYSICS OF A SPHERICAL MASS: We ask ourselves What velocity, what acceleration are "natural" to the physics of gravitating spherical mass m of radius R? From the following material

$$\begin{aligned} \text{velocity } v \equiv x_1 \quad &: \quad [x_1] = M^0 L^1 T^{-1} \\ \text{acceleration } a \equiv x_2 \quad &: \quad [x_2] = M^0 L^1 T^{-2} \\ \text{mass } m \equiv x_3 \quad &: \quad [x_3] = M^1 L^0 T^0 \\ \text{radius } R \equiv x_4 \quad &: \quad [x_4] = M^0 L^1 T^0 \\ \text{gravitational constant } G \equiv x_5 \quad &: \quad [x_5] = M^{-1} L^3 T^{-2} \end{aligned}$$

we are led to write

$$\begin{pmatrix} 0 & 0 & 1 & 0 & -1 \\ 1 & 1 & 0 & 1 & 3 \\ -1 & -2 & 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \\ k_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The `NullSpace`[*rectangular matrix*] command again provides *two* linearly independent vectors

$$\boldsymbol{k}_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{k}_2 = \begin{pmatrix} -4 \\ 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

The null space is clearly <u>closed with under linear combination</u>, and for the purposes at hand it proves more convenient to work with

$$\boldsymbol{j}_1 = \tfrac{1}{2}(\boldsymbol{k}_1 - \boldsymbol{k}_2) = \begin{pmatrix} 1 \\ 0 \\ -\frac{1}{2} \\ +\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} \quad \text{and} \quad \boldsymbol{j}_2 = 2\boldsymbol{k}_1 - \boldsymbol{k}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 2 \\ -1 \end{pmatrix}$$

which produce

$$\begin{aligned} \Pi_1 &= v^1 a^0 m^{-\frac{1}{2}} R^{\frac{1}{2}} G^{-\frac{1}{2}} \\ \Pi_2 &= v^0 a^1 m^{-1} R^2 G^{-1} \end{aligned}$$

It follows that for such a system

$$\begin{aligned} \text{characteristic velocity} \quad & v \sim \sqrt{Gm/R} \\ \text{characteristic acceleration} \quad & a \sim Gm/R^2 \sim v^2/R \end{aligned}$$

Physically the "characteristic velocity" shows up as the **escape velocity**, also as the **velocity of a satellite in low orbit**. If we set $v = c$ then $R \sim Gm/c^2$ can be understood to refer to the **radius of a blackhole** of mass m. The"characteristic acceleration" is more familiar as g.

Clearly, it would be misguided to regard

$$y = x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n} \quad \text{and (say)} \quad z \equiv y^2$$

as independent constructs, for the value/dimension of z are latent in those of y: z is *redundant* with y. Enlarging upon this remark: one says in general that physical variables of the product structure

$$\begin{aligned} y_1 &= x_1^{k_{11}} x_2^{k_{12}} \cdots x_n^{k_{1n}} \\ y_2 &= x_1^{k_{21}} x_2^{k_{22}} \cdots x_n^{k_{2n}} \\ &\vdots \\ y_p &= x_1^{k_{p1}} x_2^{k_{2p}} \cdots x_n^{k_{pn}} \end{aligned}$$

are "dimensionally dependent/independent" according as there do/don't exist exponents $\{h_1, h_2, \ldots, h_p\}$—not all of which vanish—such that

$$(y_1)^{h_1} (y_2)^{h_2} \cdots (y_p)^{h_p} \text{ is dimensionless}$$

—the idea here being that if the product *were* dimensionless one could write (say)

$$y_p = \left[(y_1)^{h_1}(y_2)^{h_2}\cdots(y_{p-1})^{h_{p-1}}\right]^{-h_p}$$

and thus render one of the y_j—here taken to be y_p—redundant. In some cases it will be possible to continue the process. A quick argument serves to establish that dimensional independence of $\{y_1, y_2, \ldots, y_p\}$ implies and is implied by the linear independence of the associated $\{\boldsymbol{k}_1, \boldsymbol{k}_2, \ldots \boldsymbol{k}_p\}$-vectors: *i.e.*, with the statement that there no not exist numbers $\{h_1, h_2, \ldots, h_p\}$—not all of which vanish—such that

$$h_1\boldsymbol{k}_1 + h_2\boldsymbol{k}_2 + \cdots + h_p\boldsymbol{k}_p = \mathbf{0}$$

Evidently we cannot expect ever to have $p > n$.

7. Buckingham's theorem. Understand $\{x_1, x_2, \ldots, x_n\}$ to refer as before to an exhaustive list of the variables, parameters and dimensioned constants that enter into the formulation of some physical theory, and let $\{\Pi_1, \Pi_2, \ldots, \Pi_p\}$ refer to some/any complete list of independent dimensionless products. Theoretical statement of the form

$$y = f(x_1, x_2, \ldots, x_n) \tag{24.1}$$

are admissible if and only if they conform to the principle of dimensional homogeneity. Statements of the design

$$\Pi = \varphi(\Pi_1, \Pi_2, \ldots, \Pi_p) \tag{24.2}$$

automatically conform to that principle (and are therefore invariant with respect to adjustment of one's system of units!). In 1914 E. Buckingham established(or at least conjectured)[5] what is in effect the converse of the preceding observation:

> **Buckingham's Π-theorem**: Every statement of the form
>
> $$y = f(x_1, x_2, \ldots, x_n)$$
>
> that conforms to the principle of dimensional homogeneity can be written as a relationship among dimensional products:
>
> $$\Pi = \varphi(\Pi_1, \Pi_2, \ldots, \Pi_p)$$

This is the upshot of what might be called the FUNDAMENTAL THEORM OF DIMENSIONAL ANALYSIS.

[5] "On physically similar systems: Illustrations of the use of dimensional analysis," Phys. Rev. **4**, 345 (1914). See also Lord Rayleigh, "The principle of similitude," Nature **95**, 66 (1915). Careful discussion of the non-trivial proof can be found in §24 of H. L. Langhaar, *Dimensional Analysis and the Theory of Models*, (1951).

The Π-theorem does tend to make familiar results look often a bit strange. If we were concerned, for example, with the dynamics of a mass point we would, instead of writing

$$F = ma$$

construct $\Pi \equiv ma/F$ and write

$$\Pi = 1$$

It should, however, be noted that $\varphi(\bullet)$ is always a function of *fewer* variables than $f(\bullet)$, and that application of the theorem leads to relations that express pure world-structure, free from the arbitrary conventions of unit selection. When we state that $\beta \equiv v/c = 0.62$ we do *not* have to report whether lengths have been measured in inches, furlongs or light years; time in seconds, hours or weeks.

> **PROBLEM 2**: Show by direct calculation that the vectors $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ that *Mathematica* supplied on page 14 do in fact lie within (and in fact span) the null space of the rectangular matrix in question.

> **PROBLEM 3**: Stars—which are held together by gravitational forces —are found on close observation to "vibrate." It seems plausible that the vibrational frequency ν depends upon the stellar diameter D, the mean stellar density ρ, and G. Use the method illustrated on pages 13–15 to construct as many dimensionless products
>
> $$\Pi(\nu, D, \rho, G)$$
>
> as you can, then use that information to deduce the necessary form of $\nu = f(D, \rho, G)$. Why do starts of the same density vibrate with the same frequency?

8. Some examples of physical interest. The examples considered thus far were designed to illustrate points of computational principle and methodology, but teach us little or nothing we did not already know about physics. Here I propose to take the methodology pretty much for granted, and to concentrate on the illustrative physics. Dimension analysis merits our attention precisely (but not only) because it *does* have the power to teach us things we *didn't* know, to provide information that is of value particularly at earliest stages of any effort to explore experimentally and to account theoretically for the phenomenon of interest.

PLANCK MASS/LENGTH/TIME It was Max Planck who first thought to ask what mass/length/time would enter most naturally into a (relativistic) **quantum theory of gravity**, a theory into which c, $\hbar$ and G enter as characteristic constants. Recalling that

$$\begin{aligned}[c] &= M^0 \quad L^1T^{-1}\\ [\hbar] &= M^1 \quad L^2T^{-1}\\ [G] &= M^{-1}L^3T^{-2}\end{aligned}$$

we do have

$$[c^{k_1}\hbar^{k_2}G^{k_3}] = M^{k_2-k_3}L^{k_1+2k_2+3k_3}T^{-k_1-k_2-2k_3}$$

and want to have

$$= M^1, \text{ else } L^1, \text{ else } T^1$$

In the first instance we write

$$\begin{pmatrix} 0 & 1 & -1 \\ 1 & 2 & 3 \\ -1 & -1 & -2 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

and by matrix inversion obtain

$$\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} +1/2 \\ +1/2 \\ -1/2 \end{pmatrix}$$

whence

$$\begin{aligned} \text{Planck mass } &= \sqrt{\hbar c/G} &&= 2.177 \times 10^{-5} \text{ g} \\ & &&= 1.302 \times 10^{+19}\text{proton masses} \end{aligned}$$

Similarly

$$\begin{aligned} \text{Planck length } &= \sqrt{G\hbar/c^3} = 1.616 \times 10^{-33}\text{cm} \\ \text{Planck time } &= \sqrt{G\hbar/c^5} = 5.391 \times 10^{-44}\text{sec} \end{aligned}$$

And from these results we infer that (for example)

$$\begin{aligned} \text{Planck density } &= c^5/G^2\hbar = 5.157 \times 10^{+93}\text{g/cm}^3 \\ \text{Planck energy } &= \sqrt{\hbar c^5/G} = 1.956 \times 10^{16}\text{erg} \end{aligned}$$

Dimensional analysis has by itself supplied the important insight that quantum gravity, whatever shape such a theory might ultimately take, can be expected to have things to say about the world only at space/time scales far, far smaller—and at energies/densities far, far larger—than those encountered in existing physics.

BALL FALLING IN VISCOUS FLUID When a spherical ball (liquid or solid, of mass m and radius r) is dropped into a fluid (gaseous or liquid, of density ρ) it achieves[6] a terminal velocity v determined in part by the viscosity μ of the fluid. We expect fluid dynamical analysis to supply a formula of the form

$$v = f(m, r, g, \rho, \mu)$$

[6] Unless it floats! That is, unless $m < \frac{4}{3}\pi r^3\rho$. Evidently the Π_2 encountered below provides a dimensionless measure of **bouyancy**.

Working from

$$\begin{aligned}
[\,v\,] &= M^0L^1T^{-1}\\
[\,m\,] &= M^1L^0T^0\\
[\,r\,] &= M^0L^1T^0\\
[\,g\,] &= M^0L^1T^{-2}\\
[\,\rho\,] &= M^1L^{-3}T^0\\
[\,\eta\,] &= M^1L^{-1}T^{-1}
\end{aligned}$$

and the requirement that $v^{k_1}m^{k_2}r^{k_3}g^{k_4}\rho^{k_5}\mu^{k_6}$ be dimensionless we are led (by *Mathematica*) to the construction of

$$\Pi_1 = \frac{v}{\sqrt{gr}}, \quad \Pi_2 = \frac{r^3\rho}{m}, \quad \Pi_3 = \frac{r^2\eta}{mv}$$

The dimensionless triple

$$\Pi_1, \quad \pi_1 \equiv \Pi_2 = \frac{r^3\rho}{m}, \quad \pi_2 \equiv (\Pi_1\Pi_3)^{-2} = \frac{m^2g}{r^3\eta^2} = \frac{mg\rho}{\eta^2}$$

> **REMARK:** The final equality results from our using the dimensionlessness of Π_2 to write $[\,m/r^3\,] = [\,\rho\,]$. The adjustment $\{\Pi_2, \Pi_3\} \rightarrow \{\pi_1, \pi_2\}$ can be considered to be the result of our replacing $\{\boldsymbol{k}_2, \boldsymbol{k}_3\}$ by certain linear combinations of those (null) vectors: we have been led by our physical good sense to over-ride decisions made by *Mathematica*.

better serves our immediate purpose, because it permits us to isolate v, writing

$$v = \sqrt{gr}\cdot\varphi(\pi_1, \pi_2)$$

A falling body experiences a velocity-dependent drag force $F_{\text{drag}} = Dv$, and the terminal velocity $u \equiv v_{\text{terminal}}$ is achieved when

$$\text{weight} = \text{drag} \quad : \quad mg = Du$$

Evidently

$$u = mg/D$$

while dimensional analysis has supplied a result that in the simplest instance reads

$$\begin{aligned}
u &= \sqrt{gr}\left(\frac{r^3\rho}{m}\right)^a\left(\frac{mg\rho}{\eta^2}\right)^b\\
&= m^{b-a}g^{b+\frac{1}{2}}r^{3a+\frac{1}{2}}\rho^{a+b}\mu^{-2b}
\end{aligned}$$

We are weakly motivated to set $b-a = b+\frac{1}{2} = 1$; *i.e.*, to set $a = -\frac{1}{2}$ and $b = +\frac{1}{2}$, in which instance

$$= \frac{mg}{\eta r} \tag{25}$$

How does this square with the physical facts? In 1850, George Stokes showed by detailed fluid dynamical analysis that the drag of a sphere of radius r moving with velocity v through a fluid with density ρ and viscosity η is (in first approximation) given by

$$F_{\text{drag}} = 6\pi\eta r \cdot v \tag{26}$$

which is the upshot of **Stokes' law**. The terminal velocity of a *falling* sphere would on this basis be given by

$$u_{\text{Stokes}} = \frac{mg}{6\pi\eta r} \tag{27}$$

which was anticipated already at (25).

The surprising ρ-independence of (25) is, as we have seen, *not* dimensionally enforced, though it is, in some sense, "dimensionally plausible."

We expect Stokes' law to break down when v becomes too great (turbulence and wave-generation become important, and bring additional variables and parameters into play) or when r becomes so small that the sphere senses the "granularity" of the fluid. Robert Millikan found that to make consistent good sense of his "oil drop" data he had to work with an improved version of (27).[7] One would expect

$$u = \frac{mg}{6\pi\eta r}\Big\{1 + \text{power series in } \pi_1 \text{ and } \pi_2\Big\}$$

to provide a rational basis for such improvements.

REMARK: To obtain the dimensionality of viscosity one has only to recall the procedure by which it is defined (see, for example, §13–10 in D. C. Giancoli, *Physics for Scientists & Engineers* (3$^{\text{rd}}$ edition, 2000)). One should, however, become familiar with the exhaustive compendium of such information that is tabulated near the end of Section F in the *Handbook of Chemistry and Physics*, and with the very clearly presented information about *many* physical/mathematical topics that can be found in *Eric Weisstein's World of Science*, on the web at

http://scienceworld.wolfram.com/

Under Physics see particularly Units & Dimensional Analysis/Dimensionless Parameters.

[7] See R. A. Millikan, *Electrons (+ and −), Protons, Neutrons, Mesotrons and Cosmic Rays* (1947), pages 90–102.

PROBLEM 4: Show *by dimensional analysis* that the centripetal force F required to constrain a mass point m to a circular orbit is proportional to m, proportional to the square of the velocity v, and inversely proportional to the orbital radius r.

PROBLEM 5: The speed u of sound in gas depends upon the pressure p and the mass density ρ. Show that $u \sim \sqrt{p}$ and $u \sim 1/\sqrt{\rho}$.

PROBLEM 6: Show that the frequency ω of any vibrational mode of a liquid drop—under the action of surface tension σ (you will need to know that $[\sigma] = [\text{energy/area}] = MT^{-2}$)—is proportional to the square root of σ, inversely proportional to the square root of the mass density ρ, and inversely proportional to the $\frac{3}{2}$ power of the diameter d.

PROBLEM 7: We conjecture that the height h of the tide caused by steady wind blowing across a lake depends upon the mean length L and depth D of the lake, the mass density ρ of the water, and the shearing stress τ of the wind on the water (you will need to know that $[\text{stress}] = [\text{force/area}]$). Show that

$$h \sim D \cdot f(L/D, \tau/g\rho D)$$

Is h therefore independent of the speed v of the wind?

PROBLEM 8: If a drop of liquid falls into a pool a small column of liquid splashes up. We expect the "splash height" h to depend upon the mass m and impact velocity of the drop, the density ρ, surface tension σ and viscosity η of the liquid, and the gravitational acceleration g. Show that

$$h = (\eta^2/g\rho^2)^{\frac{1}{3}} \cdot f(mg\rho/\eta^2, \rho v^3/\eta g, \rho\sigma^3/g\eta^4)$$

From the fact that $f(\bullet)$ has so many arguments we infer that this would be a relatively difficult system to study, either experimentally or theoretically.

9. What dimensions are primary, and how many are there? There have proceeded thus far in the unexamined presumption that physical dimension can in every instance be described

$$[\text{physical dimension}] = M^a L^b T^c$$

and that the dimensionalities M, L and T of mass/length/time are conceptually independent, irresolvable, "primary," though it was remarked near the end of §3 that an element of arbitrariness, of convention enters into this conception. I propose to explore the matter now in somewhat greater detail, and begin by expanding upon material sketched already on page 5.

By early in the 19th Century equations of the form

$$F = k\,\frac{q_1q_2}{r^2} = \frac{1}{4\pi\epsilon}\,\frac{q_1q_2}{r^2}$$

had entered the literature of physics as descriptions of the electrostatic interaction of a pair of bodies bearing electric charges q_1 and q_2. Here k (equivalently ϵ) is a phenomenological constant the value of which depends upon the specific substance (oil, water, air, ...) in which the charges are immersed. In vacuum (which was at the time a fairly rare and expensive "substance")

$$F = k_0\,\frac{q_1q_2}{r^2} = \frac{1}{4\pi\epsilon_0}\,\frac{q_1q_2}{r^2}$$

The quantification of charge issued from the quantification of electrical current (Coulomb = Ampere · second, in familiar practical units), and the quantification of current was accomplished by an electrochemical procedure (measure the rate at which silver is deposited in a standardized electroplating process). It seemed natural to assign to electric charge its own independent dimension

$$[\text{charge}] = Q$$

One then had

$$\begin{aligned}[k_0] &= M^1L^3T^{-2}Q^{-2}\\ &= [k_{\text{all material substances}}]\end{aligned}\tag{28}$$

Eventually it became evident that one might advantageously look upon k_0 as a constant of Nature, and write

$$k_{\text{material}} = (\text{correction factor})\cdot k_0$$

With that realization it became natural to make adjustments (see again page 4) so as to achieve

$$k_0 = 1 \quad (\text{dimensionless})$$

This done, one or another of the heretofore "fundamental dimensions" M, L, T, Q has been rendered reduntant with the others. Working from (28) we acquire[8]

$$Q = M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}$$

and Coulomb's law becomes

$$\begin{aligned}F &= \frac{q_1q_2}{r^2}\\ &\downarrow\\ &= \frac{1}{4\pi}\,\frac{q_1q_2}{r^2}\quad\text{in "rationalized" electrostatic units units}\end{aligned}$$

[8] It is always easier—and, since it takes two to interact, often more natural—to work with Q^2 than with Q.

That dimensional analysis leads—by different routes—to identical results whether one considers $\{M, L, T, Q\}$ or $\{M, L, T\}$ to be "fundamental" is an important point that I illustrate by

EXAMPLE: Working from

$$\begin{aligned}
[\text{electric potential } V\,] &= [\text{energy/charge}]\\
[\text{charge}] &= [d(\text{current})/dt]\\
[\text{resistance } R\,] &= [\text{potential/current}]\\
[\text{inductance } L\,] &= [\text{potential}]/[d(\text{current})/dt]\\
&= [\text{potential/charge}]\\
[\text{capacitance } C\,] &= [\text{charge/potential}]
\end{aligned}$$

we obtain

$$\begin{aligned}
[R\,] &= M^1L^2T^{-1}Q^{-2} &&= M^0L^{-1}T^0\\
[L\,] &= M^1L^2T^{\,0}Q^{-2} &&= M^0L^{-1}T^1\\
[C\,] &= M^{-1}L^{-2}T^{\,2}Q^2 &&= M^0L^{+1}T^{-1}
\end{aligned}$$

Were we to ask, within the $\{M, L, T, Q\}$ system, for dimensionless products constructable from R, L, C and a frequency ω we would proceed from $\omega^{k_1}R^{k_2}L^{k_3}C^{k_4}$ to

$$\begin{pmatrix} 0 & 1 & 1 & -1\\ 0 & 2 & 2 & -2\\ -1 & -1 & 0 & -2\\ 0 & -2 & -2 & 2 \end{pmatrix}\begin{pmatrix} k_1\\ k_2\\ k_3\\ k_4\end{pmatrix} = \mathbf{0}$$

whence (with the assistance of *Mathematica*'s `NullSpace[`*etc*`]`)

$$\boldsymbol{k}_1 = \begin{pmatrix} 1\\ 1\\ 0\\ 1\end{pmatrix}, \qquad \boldsymbol{k}_2 = \begin{pmatrix} 1\\ -1\\ 1\\ 0\end{pmatrix}$$

Were we to pose the same question within the $\{M, L, T\,\}$ system we would proceed from

$$\begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & -1 & -1 & 1\\ -1 & 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} k_1\\ k_2\\ k_3\\ k_4\end{pmatrix} = \mathbf{0}$$

and be led to the <u>same</u> set of $\boldsymbol{k}$-vectors. By either procedure we find

$$\Pi_1 = \omega RC, \qquad \Pi_2 = \frac{\omega L}{R}, \qquad \Pi_3 = \Pi_1\Pi_2 = \omega^2 LC$$

Since it is the resistance R that accounts for the dissipation in an RLC circuit, we expect *on dimensional grounds alone* to encounter

$$\begin{aligned} I(t) &= I_0 e^{-t/(RC)} && \text{in a } RC \text{ circuit} \\ I(t) &= I_0 e^{-t/(L/R)} && \text{in a } RL \text{ circuit} \\ I(t) &= I_0 \cos(t/\sqrt{LC}) && \text{in a } LC \text{ circuit} \end{aligned}$$

Fundamental to thermodynamics is the concept of **temperature** which, prior to the development of the kinetic theory of gases and of statistical mechanics, appeared to have no relationship to any mechanical concept. It appeared therefore to be natural/necessary to assign to temperature its own autonomous dimension[9]

$$[\text{temperature } T] = \theta$$

But statistical mechanics brought into focus the existence of a dimensioned constant of Nature; namely **Boltzmann's constant** k, with the dimension

$$[k] = [\text{energy/temperature}]$$

Because k is a universal constant (constant of Nature) we are free to use

$$\mathfrak{T} \equiv kT \quad : \quad [\mathfrak{T}] = [\text{mechanical energy}] = ML^2T^{-2}$$

rather than T itself to quantify temperature. The discovered existence of k has here been used to contract the system of fudamental dimensions:

$$\{M, L, T, \theta\} \xrightarrow[k]{} \{M, L, T\}$$

Contemporary physics supplies also other universal constants; namely c, $\hbar$, the quantum of charge e, and Newton's G, and those permit us to extend the contraction process. For example: relativity recommends that we use

$$x^0 \equiv ct \quad : \quad [x^0] = [\text{length}] = L$$

rather than t itself to quantify time. And the definition $\hbar/mc$ of the **Compton length** recommends that we use

$$\mu \equiv (c/\hbar)m \quad : \quad [\mu] = L^{-1}$$

rather than m itself to quantify mass. At this point we have achieved

$$\{M, L, T\} \xrightarrow[c]{} \{M, L\} \xrightarrow[\hbar]{} \{L\}$$

[9] Do not confuse the <u>temperature T</u> with the <u>dimension T of time</u>.

Relativistic quantum theory provides no "natural length," so we have come here to the end of the line.[10]

If, however, we were to bring G into play then we could use

$$\ell \equiv \frac{\text{length } x}{\text{Planck length } \sqrt{G\hbar/c^5}} \quad : \quad [\,\ell\,] \textbf{ dimensionless}$$

rather than x itself to quantify length. Were we to adopt such a procedure then all physical variables would be rendered dimensionless, and dimensional considerations would (now in physics as standardly in mathematics) place **no constraint at all** on statements of the form[11]

$$y = f(x_1, x_2, \ldots, x_n)$$

The principle of dimensional homogeneity (the Π-theorem (24.2)) has collapsed into useless triviality.

If we abandon reference to G (on grounds that gravitational effects are typically irrelevant to the physics of interest, and the Planck length so absurdly small) then we would have

$$[\,\text{every physical variable}\,] = L^{\text{some power}}$$

and the argument presented on page 8 would simplify markedly: in place of (14) we have simply

$$\text{length units} = B \text{ new length units}$$

If $[X] = L^b$ then the measured value x of X then becomes $\bar{x} = xB^b$. In place

[10] Notice that no use has been or will be made of e, for the interesting reason that the

$$\text{fine structure constant } \alpha \equiv \frac{e^2}{\hbar c} \text{ is } \textbf{dimensionless}$$

so $[e^2]$ is redundant with $\{[\hbar], [\,c\,]\}$.

[11] See again equations (16) = (24.1).

of (18) we have

$$\begin{aligned}
y \;&\longmapsto \bar{y} \;= y\;B^{b} \;\;\equiv K\;\; y\\
x_1 &\longmapsto \bar{x}_1 = x_1 B^{b_1} \equiv K_1 x_1\\
x_2 &\longmapsto \bar{x}_2 = x_2\, B^{b_2} \equiv K_2 x_2\\
&\vdots\\
x_n &\longmapsto \bar{x}_n = x_n B^{b_n} \equiv K_n x_n
\end{aligned}$$

The statement (19) of the principle of dimensional homogeneity can now be notated

$$B^b f(x_1, x_2, \dots x_n) = f(B^{b_1}x_1, B^{b_2}x_2, \dots, B^{b_n}x_n)$$

If—following in the footsteps of Euler—we differentiate with respect to B and then set $B = 1$ we obtain

$$b \cdot f(x_1, x_2, \dots x_n) = \Big\{b_1 x_1 \frac{\partial}{\partial x_1} + b_2 x_2 \frac{\partial}{\partial x_2} + \cdots + b_n x_n \frac{\partial}{\partial x_n}\Big\} f(x_1, x_2, \dots x_n)$$

And if, in particular, $y = x_1^{k_1} x_2^{k_2} \cdots x_2^{k_2}$ then we must have (compare (20.2))

$$b_1 k_1 + b_2 k_2 + \cdots + b_n k_n = b$$

which imposes only a single condition on the exponents $\{k_1, k_2, \dots, k_n\}$.

SIMPLE PENDULUM REVISITED ONCE AGAIN: Proceeding once again from the conjecture that it is possible to write

$$\text{period} = (\text{mass of bob})^{k_1} \cdot (\text{length of rod})^{k_2} \cdot (g)^{k_3}$$

we have

$$L^1 = L^{-k_1} L^{k_2} L^{-k_3}$$

giving

$$1 + k_1 - k_2 + k_3 = 0$$

whence—for *all* $\{k_1, k_3\}$—

$$\text{period} = (\text{mass of bob})^{k_1} \cdot (\text{length of rod})^{1+k_1+k_3} \cdot (g)^{k_3}$$

We do recover the familiar result if we set $k_1 = 0$ and $k_3 = -\frac{1}{2}$ but now have no particular reason to do so!

Evidently *dimensional analysis becomes a weaker and ever weaker tool as the size of the system of "fundamental dimensions" is progressively contracted*. Which is counterintuitive, for it is the advance of physics—the discovery of fundamental relations[12] and of universal constants—that fuels that contraction.

[12] Think of $F = ma$, which declares measurements performed with spring scales to be redundant with measurements performed with measuring rods and clocks.

10. Application of dimensional analysis to the design of models. The engineers and physicists responsible for the design of large/complex/expensive structures (boat hulls, air frames, bridges, telescope and accelerator components) often study scaled-down **dynamical models**[13] to gain confidence in and to fine-tune their designs. It is, however, intuitively evident that cinematic footage of a toy forest fire, or a toy storm at sea, will not look convincingly like the real thing, but becomes somewhat more convincing when projected in slow motion. It is not immediately obvious what real-world information can be inferred from study of (say) the drag on toy models of ships. And it would clearly be frivolous to contemplate construction of a functional ten-meter model of the sun. Or of a galaxy. Or—to compound the absurdity—of a hydrogen atom! Clearly, the design of informative models is subject to some severe constraints-in-principle. Historically, it was a desire to clarify the source of those constraints—to construct of an orderly "theory of models"—that served as the primary motivation for the *invention* of dimensional analysis.[14] I will illustrate the points at issue by discussion of a realistic

DRAG ON A YACHT Our assignment is to determine how hull-shape affects the drag on a 20-meter yacht. This we propose to do by measuring the drag on one-meter models of similar shape and mass distribution. We expect that detailed dynamical analysis (if it could be carried out) would culminate in a formula of the form

$$\text{drag } D = f(v, \ell, \eta, \rho, g)$$

where v refers to the yacht's speed, ℓ to its characteristic length, η to the viscosity of water, ρ to the density of water, and g enters because the yacht produces a wake that undulates energetically in the earth's gravitational field. Arguing as on page 20, we have

$$\begin{aligned}
[\,D\,] &= M^1L^1T^{-2}\\
[\,v\,] &= M^0L^1T^{-1}\\
[\,\ell\,] &= M^0L^1T^0\\
[\,\eta\,] &= M^1L^{-1}T^{-1}\\
[\,\rho\,] &= M^1L^{-3}T^0\\
[\,g\,] &= M^0L^{-1}T^{-2}
\end{aligned}$$

and from the requirement that $D^{k_1}v^{k_2}\ell^{k_3}\eta^{k_4}\rho^{k_5}g^{k_6}$ be dimensionless

[13] Architects also construct *static* models to study æsthetic aspects of their designs, which is quite a different matter.

[14] This effort seems to have originated with Maxwell ("On the mathematical classification of physical quantities," Proc. London Math. Soc. **3**, 224 (1871)), but see also the papers by Lord Rayleigh and E. Buckingham cited previously.[5]

we are to write

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & -1 & -3 & 1 \\ -2 & -1 & 0 & -1 & 0 & -2 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \\ k_5 \\ k_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and thus (with the assistance again of *Mathematica*'s `NullSpace` command) to the construction of dimensionless expressions

$$\begin{aligned} \Pi_1 &= \frac{D}{\rho v^2 \ell^2} \\ \Pi_2 \equiv \textbf{Reynolds' number } R &= \frac{v\ell\rho}{\eta} \\ \Pi_3 \equiv \textbf{Froud number } F &= \frac{v^2}{g\ell} \end{aligned}$$

We expect therefore to have

$$D = \rho v^2 \ell^2 \cdot f(R, F)$$

The physical argument that leads from hull-shape to the specific design of $f(\bullet,\bullet)$, but we can proceed without that information on strength of the **assumption** that the function in question pertains to *all hulls of the same shape*, irrespective of size.[15] Let numbers $\{D, v, \ell, \eta, \rho, g\}$ refer to the yacht, and (in the same units) numbers

$$\begin{aligned} D' &= K_D \cdot D \\ v' &= K_v \cdot v \\ \ell' &= K_\ell \cdot \ell \quad : \quad \text{we have agreed to set } K_\ell = \tfrac{1}{20} \\ \eta' &= K_\eta \cdot \eta \\ \rho' &= K_\rho \cdot \rho \\ g' &= K_g \cdot g \end{aligned}$$

refer to the model. If the model is to mimic the behavior of the full scale yacht then

$$\begin{aligned} \Pi_1{}' &= \frac{K_D}{K_\rho (K_v)^2 (K_\ell)^2} \cdot \Pi_1 \text{ must} = \Pi_1 \\ R' &= \frac{K_v K_\ell K_\rho}{K_\eta} \cdot R \qquad \text{must} = R \\ F' &= \frac{(K_v)^2}{K_g K_\ell} \cdot F \qquad \text{must} = F \end{aligned}$$

[15] This assumption cannot be maintained under all possible circumstances: for miniature hulls we expect surface tension—of which we have taken no account—to contribute importantly to drag.

If both yacht and model float in water, and experience the same gravity, then necessarily $K_\eta = K_\rho = K_g = 1$, and we must have

$$K_D = (K_v K_\ell)^2, \quad K_v K_\ell = 1 \quad \text{and} \quad K_v^2 = K_\ell$$

which taken together imply $K_v = K_\ell = K_D = 1$: the model must be the same size as the yacht itself! But if the hull-shape is designed to minimized the effect of viscosity then we can drop the second condition, and obtain

$$\begin{aligned} K_v &= (K_\ell)^{\frac{1}{2}} \\ K_D &= (K_\ell)^3 \end{aligned}$$

If $K_\ell = \frac{1}{20}$ then the model should be propelled at 22% of the intended speed of the yacht, and the measured drag will have to be multiplied by 8000 to give the predicted drag on the yacht. If, on the other hand, wake production is negligible (as it is for submarines) then we can drop the third condition, and obtain

$$\begin{aligned} K_v &= (K_\ell)^{-1} \\ K_D &= 1 \end{aligned}$$

The model, if propelled at 20 times the intended speed of the yacht (submarine), will experience the *same* drag as the full-scale vessel (but not really, for we have taken no account of turbulence).

MODELS OF HYDROGEN ATOMS A good approximation to the physically correct theory of atoms arises when one looks quantum mechanically ($\hbar$) to the electromagnetic interaction (e^2) of electrons with one another and with the much more massive nucleus. In the simplest instance one has the hydrogen atom, with its single electron. Let R—call it the "Bohr radius"—refer to the natural "atomic length.". From $\{R, m, e^2, \hbar\}$ one can assemble but a single dimensionless expression; namely

$$\Pi = Rme^2/\hbar^2$$

One therefore expects to have

$$\text{Bohr radius } R = \frac{\hbar^2}{me^2} \approx 0.529189379 \times 10^{-10}\,\text{m}$$

To make an enlarged functional model of the hydrogen atom one has—since e^2 and $\hbar$ are unalterable constants of Nature—no alternative but to proceed $m \to m' = K_m \cdot m \ll m$. But that is *impossible, since Nature provides no charged particle less massive than an electron.*

One can draw several general lessons from the preceding examples:

- It is sometimes **not possible** to dilate variables/parameters in such a way as to preserve the values of *all* the independent dimensionless expressions $\{\Pi_1, \Pi_2, \ldots, \Pi_q\}$. One has then to abandon the least important of the Π's, and to make do with *approximate* models.
- The intrusion of natural constants (c, e^2, k, $\hbar$, G, particle masses, mole numbers) into the construction of $\{\Pi_1, \Pi_2, \ldots, \Pi_q\}$ tends—since natural constants are not susceptible to adjustment—to inhibit the construction of models. This is a difficulty confronted more often by physicists (who for this reason cannot expect to construct functional scale models of the sun) than by engineers, whose equations seldom contain natural constants.
- The equations of interest to engineers do, on the other hand, make frequent reference to the properties of materials (density, surface tension, elastic moduli, conductivity, *etc.*) and in Nature's Stockroom the variability of those is in most instances severely limited. It would, for instance, be frivolous to ask the model maker to use a material that is ten times denser than iron, but one fifth as stiff!

11. Some illustrative biomechanical consequences of scale.[16] Let ℓ refer to the characteristic length of an animal—an animal, let us say, with legs. Specifically, we might set

$$\ell = \big[\text{volume of an equivalent mass of water}\big]^{\frac{1}{3}}$$

We are informed that the largest blue whale (the largest animal that has ever lived, so far as is known) is about 10^{21} times more massive tahn the smallest microbe, which suggests that $\ell_{\text{max}} \approx 10^7 \ell_{\text{min}}$. But the creatures at those extremes both live (leglessly) in aquatic environments, and the smallest swimmers inhabit a world that is—for reasons that I will have occasion to explore[17]—profoundly unlike that experienced by the terrestrial creatures to which we have agreed to restrict our attention. For the latter we expect to have something like

$$\ell_{\text{terrestrial max}} \approx 1\,\text{m} \approx 10^4 \ell_{\text{terrestrial max}}$$

We expect the rate at which a terrestrial creature can unload metabolically generated heat to be roughly proportional to its surface area; *i.e.*, to go as ℓ^2. Relatedly, we expect the rate at which such a creature can take in oxygen to be proportional to the surface area of its lungs: to go again as ℓ^2. And we expect its strength to be proportional to the cross sections of its bones and muscles: ℓ^2 again. For all these reasons we expect large creatures to be relatively less

[16] The following discussion draws heavily upon material presented in Chapter 1 of J. Maynard Smith, *Mathematical Ideas in Biology* (1971) and in the wonderfully detailed and beautifully illustrated monograph *On Size and Life*, by Thomas A. McMahon & John Tyler Bonner (1983).

[17] In the meantime, see E. M. Purcell's classic "Life at small Reynolds numbers," AJP **45**, 3 (1977).

powerful, less strong. We expect smaller creatures to have relatively more slender legs, lighter musculature. We are not surprised that ants can lift many times their own weight, while horses can carry only a fraction of their weight.

Look to the maximal running speed of animals with legs. Working from

$$\text{power} = (\text{force}) \cdot (\text{velocity})$$

and taking force to be determined by the maximal strength of bones and muscles, we obtain

$$\ell^2 \sim \ell^2 v$$

and conclude that in leading approximation *all animals have the same top speed*. This surprising conclusion is borne out for animals sized between rabbits and horses. The refined problem would be to explain why both elephants and bugs run more slowly than cheetahs (though in terms of body length per unit time many bugs are much *faster* than cheetahs!).

When running uphill an animal must do work agtainst the gravitational field at a rate proportional to

$$(\text{mass}) \cdot g \cdot v \sim \ell^3 v$$

When set equal to the available power this gives

$$v_{\text{max uphill}} \sim 1/v$$

This conclusion is supported by the facts: hills up which horses are obliged to walk are hills up which little dogs are able to run.

Because the volume of the metabolic fuel containers (stomach and lungs) increases as ℓ^3 while the metabolic rate of energy production (power) increases less rapidly (as ℓ^2), we expect larger animals to be able to work longer between meals, and larger aquatic mammals to be able to dive relatively deeper.

Look to FIGURE 1. The jumper invests energy $W = (\text{force}) \cdot d \sim \ell^2 \cdot \ell^1$ in executing its jump, and rises to a height given by $h = W/mg \sim \ell^2 \cdot \ell^1/\ell^3 = \ell^0$. Thus are we led to expect all animals to jump to roughly the same height, fleas to be able to jump as high as dogs, dogs as high as kangaroos.

Many animals adjust their gait when they want to go faster, or faster still, adopting modes of locomotion in which their feet spend progressively less time on the ground, more time off the ground. Defining the "jumpiness" J by

$$J \equiv \frac{\text{air time}}{\text{ground time}}$$

he is able to show that in leading approximation one might expect to have

$$\frac{\text{ground time} + \text{air time}}{\text{ground time}} = 1 + J \sim v^2/\ell$$

which gets larger as the the speed v increases, smaller as the animal gets larger, in qualitative conformity with the familiar facts (think of rabbits, human runners, horses, elephants).

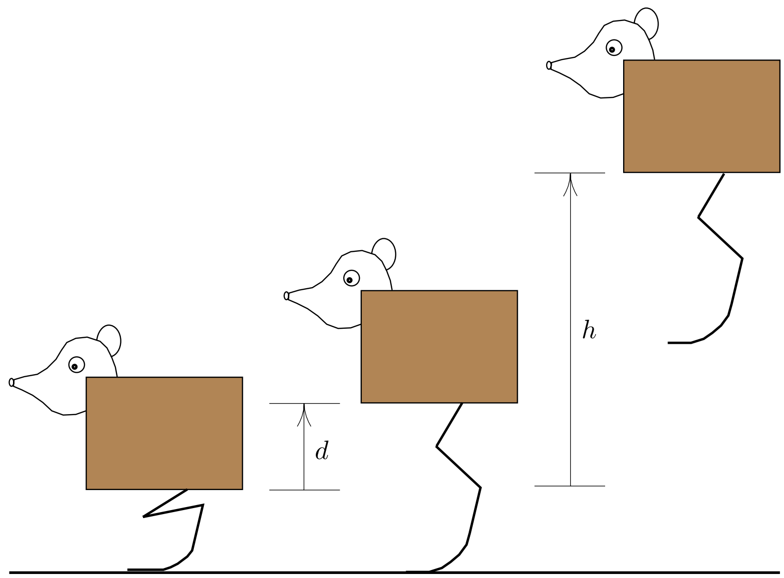

FIGURE 1: *The figure, read from left to right, shows a jumper in full prepatory crouch*; *the jumper with legs fully extended*; *the jumper at the top of its leap. The figure has been adapted from* FIGURE 5 *in J. Maynard Smith's little monograph.*[16]

PROBLEM 9: Argue why it is that sufficiently small animals are able to walk up walls and across the ceiling.

PROBLEM 10: Argue why it is that heart rate can be expected to decrease with increasing size.

12. Life at small Reynolds number. We learned from Newton that—contrary to what Aristotle imagined to be the case—force is required to *accelerate* objects that move in empty space, but *no force is required to maintain uniform motion*. But steady velocity-dependent force *is* required to maintain the steady motion of an object in a fluid environment, even the absence of viscosity effects. For this reason: massive fluid elements must be accelerated to "get out of the way," to create a co-moving cavity. In this respect, particle dynamics within a fluid appears on its face to be more Aristotelian than Newtonian.

Thee viscosity-independent force in question is called the "inertial force," and by dimensional analysis we expect to have

$$F_{\text{inertial}} = (\text{geometrical factor}) \cdot (\rho\ell^3)(v^2/\ell) \sim \rho v^2 \ell^2$$

where ρ refers to the density of the fluid, ℓ to a length characteristic of the object, and the "geometrical factor" refers in a complicated way to its shape.

Viscosity is defined by an operational procedure (see again pages 19 & 20) that entails $[\eta^2] = M^2L^{-2}T^{-2} = (ML^{-3})(MLT^{-2}) = [\text{density}]\cdot[\text{force}]$ so we expect to have

$$F_{\text{viscous}} = (\text{geometrical factor})\cdot\eta^2/\rho$$

Therefore

$$\frac{F_{\text{inertial}}}{F_{\text{viscous}}} \sim \frac{\rho^2v^2\ell^2}{\eta^2} = (\text{Reynolds number } R)^2$$

which is to say[18]

$$R \sim \sqrt{\frac{F_{\text{inertial}}}{F_{\text{viscous}}}}$$

Viscosity η and density ρ both refer to properties of the fluid. In practice it often proves convenient to work instead with a single conflated parameter called the "kinematic viscosity"

$$\nu = \eta/\rho$$

which has the dimensionality of area/time. In this notation Reynolds' number becomes

$$R = \ell v/\nu$$

For water

$$\nu_{\text{water}} \approx 10^{-2}\,\text{cm}^2/\text{sec}$$

For a swimming person we might reasonably set $\ell \approx 50\,\text{cm}$ (the diameter of a 144-pound sphere of water) and $v \approx 100\,\text{cm/sec}$, which would give

$$R_{\text{swimmer}} \approx 5\times 10^5$$

For a guppy we might take $\ell \approx 5\times 10^{-1}$ cm and $v \approx 10\,\text{cm/sec}$, giving

$$R_{\text{guppy}} \approx 5\times 10^2$$

For a microbe it is, according to Purcell,[17] reasonable to set $\ell \approx 10^{-4}$ cm (one micron) and $v \approx 3\times 10^{-3}$ cm/sec, which give

$$R_{\text{microbe}} \approx 3\times 10^{-5}$$

If the force that is propelling an object through a viscous fluid is suddenly turned off, and if also $R \ll 1$, then we can on dimensional grounds expect the object to come to rest with a characteristic

$$\text{stopping time} \sim \tau \equiv \ell^2/\nu = R\ell/v$$

and during that time to travel a characteristic

$$\text{stopping distance} \sim \sigma \equiv v\tau = R\ell$$

[18] I cannot account for the fact that many/most authors—including Purcell—neglect to include the $\sqrt{\quad}$.

Our microbe would coast only a few hundred thousandths of a body-length! To share the swimming experience of a microbe you would have to swim in a pool filled with a fluid 10^{10} times more viscous than water. It is intuitively evident that whatever the "cyclically sequenced contortions" you might devise as a propulsion strategy, it is unlikely that you would be moved to call that exercise "swimming." "Corkscrewing/snaking" would appear—intuitively—to be more apt, and the microbes themselves seem to agree. Microbes—brainless though they be—have evolved strategies that enable them to move beyond regious at risk of becoming over-grazed, and to move beyond the expanding sphere of their diffusing waste products. For discussion of these and other aspects of "life at very small Reynolds number" I do urge my readers to have a look at Purcell's little article:[17] it is easy to read, but highly informative.